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# Optimal Control of Double Inverted Pendulum Using LQR Controller

Sandeep Kumar Yadav
*M.Tech (2nd), ICE Dept.*
*Dr. B.R.A. NIT Jalandhar*
*Punjab, India*

Sachin Sharma
*M.Tech (2nd), ICE Dept.*
*Dr. B.R.A. NIT Jalandhar*
*Punjab, India*

Mr. Narinder Singh
*Associate Prof. ICE Dept.*
*Dr. B.R.A. NIT Jalandhar*
*Punjab, India*

***Abstract*—Double Inverted Pendulum is a nonlinear system, unstable and fast reaction system. Double Inverted Pendulum is stable when its two pendulums allocated in vertically position and have no oscillation and movement and also inserting force should be zero. The aim of the paper is to design and performance analysis of the double inverted pendulum and simulation of Linear Quadratic Regulator (LQR) controller. Main focus is to introduce how to built the mathematical model and the analysis of it's system performance, then design a LQR controller in order to get the much better control. Matlab simulations are used to show the efficiency and feasibility of proposed approach.**

***Keywords-Double inverted pendulam; Matlab; Linear-quardratic regulator; system performance;***

## I. Introduction

Pendulum system is one of the classical questions of study and research about non-linear control and a suitable criterion for harnessing the mechanical system. Double inverted pendulum is a typical model of multivariable, nonlinear, essentially unsteady system, which is perfect experiment equipment not only for pedagogy but for research because many abstract concepts of control theory can be demonstrated by the system-based experiments. Because of its non-linearity and instability is a useful method for testing control algorithms (PID controls, neural network, fuzzy control, genetics algorithm, etc.). In 1972, two researchers from the control domain succeeded, using an analogue computer, in controlling an inverted pendulum in standing position on a cart, which was stabilized by horizontal force [1]. There are various types of inverted pendulum such as, the simple inverted pendulum, the rotary inverted pendulum, double inverted pendulum, the rotatory double inverted pendulum [2]. The research on such a complex system involves many important theory problems about system control, such as nonlinear problems, robustness, ability and tracking problems. Therefore, as an ideal example of the study, the inverted pendulum system in the control system has been universal attention. And it has been recognized as control theory, especially the typical modern control theory research and test equipment. So it is not only the best experimental tool but also an ideal experimental platform. The research of inverted pendulum has profound meaning in theory and methodology, and has valued by various countries' scientists [1].

The double inverted pendulum is a non-linear, unstable and fast reaction system. This system consists of two inverted pendulums assembled on each other, as shown in fig. 1, and mounted on a cart that can be controlled and stabilized through applying the force F.

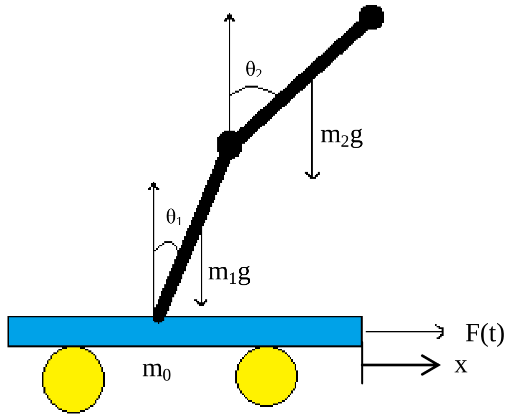


Fig. 1: Model of double inverted pendulum system

Linear quadratic optimal control theory( LQR) is the most important and the most comprehensive of a class of optimization-based synthesis problem to obtain the performance index function for the quadratic function of the points system, not only taking into account both performance requirements, but also taking into account the control energy requirements.

In this paper we have modeled and designed a double inverted pendulum and then we have studied its response so that it should meet our performance and requirement and to achieve this we have designed a LQR controller.

## II. MATHEMATICAL MODELING OF DOUBLE INVERTED PENDULAM

| Symbol | Parameter | Value | Unit |
|---|---|---|---|
| $m_0$ | Mass of the cart | 0.8 | Kg |
| $m_1$ | Mass of the first pendulum | 0.5 | Kg |
| $m_2$ | Mass of the second pendulum | 0.3 | Kg |
| $L_1$ | Length of the first pendulum | 0.3 | m |
| $L_2$ | Length of second pendulum | 0.2 | m |
| J | Inertia of the pendulum ($j_1$=$j_2$) | 0.006 | Kg $m^2$ |
| g | Canter of gravity | 9.8 | $m/s^2$ |

Assuming the parameters of double inverted pendulum system as shown in table 1.

To derive its equations of motion, one of the possible ways is to use Lagrange equations [4], which are given as follows.

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0 \;; \quad i=0,1,2,\ldots\ldots\ldots n \qquad (1)$$

where L = T - V is a Lagrangian, Q is a vector of generalized forces (or moments) acting in the direction of generalized coordinates $q$ and not accounted for in formulation of kinetic energy T and potential energy V.

Table 1: Parameters of double inverted Pendulum system

The kinetic energy of the system, according the fig.1, is composed of the cart, the first pendulum and the second pendulum kinetic energies [11]. The cart kinetic energy is:

$$T_o=\frac{1}{2}m_0\dot{x}^2 \qquad (2)$$

The first pendulum kinetic energy is equal to the sum of three horizontal, vertical and rotational energy of pendulum.

$$T_1=\frac{1}{2}m_1[\dot{x}+\dot{\theta}_1 l_1 cos\theta_1]^2+\frac{1}{2}m_1\dot{\theta}_1^{\,2} l_1^{\,2} sin^2\theta_1+\frac{1}{2}J_1\dot{\theta}_1^{\,2} \qquad (3)$$

The kinetic energy of the second pendulum is also identical to the first pendulum. (4)

$$T_2=\frac{1}{2}m_2[\dot{x}+\dot{\theta}_1 l_1 cos\theta_1+l_2\theta_2 cos\theta_2]^2 +\frac{1}{2}m_2[\dot{\theta}_1 L_1 sin\theta_1+\dot{\theta}_2 L_2 sin\theta_2]^2+\frac{1}{2}J_2\dot{\theta}_2^{\,2}$$

The system kinetic energy is gained from the sum of three equations.

$$T=T_o+T_1+T_2$$

$$T=\frac{1}{2}m_0\dot{x}^2+\frac{1}{2}m_1[\dot{x}+\dot{\theta}_1 l_1 cos\theta_1]^2+\frac{1}{2}m_1\dot{\theta}_1^{\,2} l_1^{\,2} sin^2\theta_1+\frac{1}{2}J_1\dot{\theta}_1^{\,2}+\frac{1}{2}m_2[\dot{x}+\dot{\theta}_1 l_1 cos\theta_1+l_2\theta_2 cos\theta_2]^2+\frac{1}{2}m_2[\dot{\theta}_1 L_1 sin\theta_1+\dot{\theta}_2 L_2 sin\theta_2]^2+\frac{1}{2}J_2\dot{\theta}_2^{\,2} \qquad (5)$$

By simplification of the relationship (5) we have:

$$T=\frac{1}{2}(m_0+m_1+m_2)\dot{x}^2+\frac{1}{2}(m_1 l_1^{\,2}+m_2 L_1^{\,2}+J_1)\dot{\theta}_1^{\,2}+\frac{1}{2}(m_2 l_2^{\,2}+J_2)\dot{\theta}_2^{\,2}+(m_1 l_1+m_2 L_1)\dot{x}\dot{\theta}_1 cos\theta_1+m_2 l_2\dot{x}\dot{\theta}_2 cos\theta_2+m_2 L_1 l_2 \cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 \qquad (6)$$

Now we calculate the potential energy of the system separately for three parts of the system. The cart potential energy is zero.

$$V_o=0 \qquad (7)$$

The potential energy of the two pendulums will be as the following respectively:

$$V_1=m_1 g l_1 cos\theta_1 \qquad (8)$$

$$V_2=m_2 g(L_1 cos\theta_1+l_2 cos\theta_2) \qquad (9)$$

The system potential energy is gained from the sum of three (7), (8) and (9) equations.

$$V=V_0+V_1+V_2 \qquad (10)$$

$$V=(m_1 l_1+m_2 L_1)g cos\theta_1+m_2 l_2 g\, cos\theta_2$$

Putting the (6) and the (10) equations in the Lagrange equation we have:

$$L=\frac{1}{2}(m_0+m_1+m_2)\dot{x}^2+\frac{1}{2}(m_1 l_1^{\,2}+m_2 L_1^{\,2}+J_1)\dot{\theta}_1^{\,2}+\frac{1}{2}(m_2 l_2^{\,2}+J_2)\dot{\theta}_2^{\,2}+(m_1 l_1+m_2 L_1)\dot{x}\dot{\theta}_1 cos\theta_1+m_2 l_2\dot{x}\dot{\theta}_2 cos\theta_2+m_2 L_1 l_2 \cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2-(m_1 l_1+m2L1gcos\theta 1-m2l2g\, cos\theta 2 \qquad (11)$$

Differentiating the Lagrangian by $q$ and $\dot{q}$ yields Lagrange equation (1) as:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=u \qquad (12)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=0 \qquad (13)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}=0 \qquad (14)$$

According to the equation (12), there is an external force of u (F (t)) only in X direction. Then we apply the equations (12), (13) and (14) separately on the equation (11).

$$(m_0+m_1+m_2)\ddot{x}+(m_1 l_1+m_2 L_1)\ddot{\theta}_1 cos\theta_1+m_2 l_2\ddot{\theta}_2 cos\theta_2-(m_1 l_1+m_2 L_1)\dot{\theta}_1^{\,2} sin\theta_1-m_2 l_2\dot{\theta}_2^{\,2} sin\theta_2=u \qquad (15)$$

$$(m_1 l_1^{\,2}+m_2 L_1^{\,2}+J_1)\ddot{\theta}_1+(m_1 l_1+m_2 L_1)\ddot{x}\, cos\theta_1+m_2 L_1 l_2 \cos(\theta_1-\theta_2)\,\ddot{\theta}_2+m_2 L_1 l_2 \sin(\theta_1-\theta_2)\,\dot{\theta}_2^{\,2}-(m_1 l_1+m_2 L_1)g sin\theta_1=0 \qquad (16)$$

$$\left(m_2 l_2^{\ 2} + J_2\right)\ddot{\theta_2} + m_2 l_2 \ddot{x}\, cos\theta_2 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta_1} - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\,\dot{\theta_1}^{\ 2} - m_2 l_2 g\, sin\theta_2 = 0 \quad (17)$$

Let we assume that

$a_0 = (m_0 + m_1 + m_2)$

$a_1 = m_1 l_1 + m_2 L_1$

$a_2 = m_1 l_1^{\ 2} + m_2 L_1^{\ 2} + J_1$

$a_3 = m_2 l_2$

$a_4 = m_2 L_1 l_2$

$a_5 = m_2 l_2^{\ 2} + J_2$

$$a_0\, \ddot{x} + a_1\, \ddot{\theta_1}\, cos\theta_1 + a_3\, \ddot{\theta_2} cos\theta_2 - a_1 \dot{\theta_1}^{\ 2}\, sin\theta_1 - a_3 \dot{\theta_2}^{\ 2} sin\theta_2 = \text{u} \quad (18)$$

$$a_1\, \ddot{x}\, cos\theta_1 + a_2 \ddot{\theta_1} + a_4\, \ddot{\theta_2}\, \cos(\theta_1 - \theta_2) + a_4 \dot{\theta_2}^{\ 2} \sin(\theta_1 - \theta_2) - a_1 \text{g} \sin\theta_1 = 0 \quad (19)$$

$$a_3\, \ddot{x} \cos\theta_2 + a_4 \ddot{\theta_1} \cos(\theta_1 - \theta_2) + a_5 \ddot{\theta_2} - a_4 \sin(\theta_1 - \theta_2)\, \dot{\theta_1}^{\ 2} - a_3\, g\, sin\theta_2 = 0 \quad (20)$$

The controller can only work with the linear function so these set of equation should be linearized about $\theta_1 = \theta_2 = 0$, $\sin\theta_1 = \Phi_1$, $\sin\theta_2 = \Phi_2$ and $\sin(\theta_1 - \theta_2) = \Phi_1 - \Phi_2$, $\cos(\theta_1 - \theta_2) = 1$ and $\cos\theta_1 = 1, \cos\theta_2 = 1$ and $\dot{\theta}_1 = \dot{\theta}_2 = 0$ and make u to represent the controlled object with the input force f(t).

$$a_0\, \ddot{x} + a_1\, \ddot{\phi}_1 + a_3\, \ddot{\phi}_2 = \text{u}$$
$$a_1\, \ddot{x} + a_2 \ddot{\phi}_1 + a_4\, \ddot{\phi}_2 - a_1 \text{g}\, \Phi_1 = 0$$
$$a_3\, \ddot{x} + a_4 \ddot{\phi}_1 + a_5 \ddot{\phi}_2 - a_3\, g\, \Phi_2 = 0$$

First to be selected the state variable, we assume the X as the cart displacement $\dot{X}$ as the displacement velocity, $\theta_1$ the first pendulum angle and $\dot{\theta}_1$ its angular velocity, $\theta_2$ the second pendulum angle and $\dot{\theta}_2$ its angular velocity, all as the state variables of the double inverted pendulum system. We obtain the following state space equation.

$$\dot{X} = \text{AX+BU and } \text{Y=CX+DU}$$

$$\text{A}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.0495 & 0 & -0.5813 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 46 & 0 & -6.6086 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -35.68 & 0 & 41.20 & 0 \end{bmatrix} \quad \text{B}=\begin{bmatrix} 0 \\ 0.8459 \\ 0 \\ -2.5204 \\ 0 \\ -0.2998 \end{bmatrix}$$

$$\text{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## III. CHARACTERISTICS ANALYSIS OF SYSTEM

After obtaining the mathematical model of the system features, we need to analyze the stability, controllability and observability of system's in order to further understand the characteristics of the system [5~7].

### A. Stability Analysis

If the closed-loop poles are all located in the left half of "s" plane, the system must be stable, otherwise the system instability. In MATLAB, to strike a linear time-invariant system, the characteristic roots can be obtain by eig (a,b) function. According to the sufficient and necessary conditions for stability of the system, we can see the inverted pendulum system is unstable.

### B. Controllability Analysis

Linear time-invariant controllability systems necessary and sufficient condition is:

$$\text{rank}[\text{B AB } \text{A}^2\text{B}\ldots\ldots\text{A}^{n-1}\text{B}]=\text{n}$$

The dimension of the matrix A is n. In MATLAB, the function ctrb(a,b) is used to test the controllability of matrix ,through the calculation we can see that the system is status controllable.

### C. Observability Analysis

Linear time-invariant observability systems necessary and sufficient condition is:

$$\text{rank}[\text{C CA } \text{CA}^2\ldots\ldots\text{CA}^{n-1}]^T=\text{n}$$

In MATLAB, the function obsv(a,b) is used to test the observability of matrix ,through the calculation we can see that the system is status considerable.

## IV. DESIGN OF LQR CONTROLLER

In this section, we will present the design of the LQR controller and then we will the show the simulation of the system.

### A. LQR Controller

The basic principles of LQR linear quadratic optimal control, by the system equation:

$$\dot{x}\text{ (t)=A x(t)+B u(t)}$$

$$\text{Y(t)=C x(t)}$$

And the quadratic performance index functions:

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)\, dt$$

Q is a positive semi-definite matrix, R is positive definite matrix. If this system is disturbed and offset the zero state, the control u can make the system come back to zero state and J is minimal at the same time [8]. Here, the control value u is called optimal control the control signal should be:

$$u(t) = -R^{-1} B^T P(t) x(t) = -Kx(t) \quad (21)$$

Where P(t) is the solution of Riccati equation, K is the linear optimal feedback matrix. Now we only need to solve the Riccati equation (21):

$$PA + A^T - PBR^{-1}P + Q = 0 \quad (22)$$

Then we can get the value of P and K.

$$K=R^{-1}B^{T}P=[k_1,k_2,k_3,k_4,k_5,k_6]^{T}$$

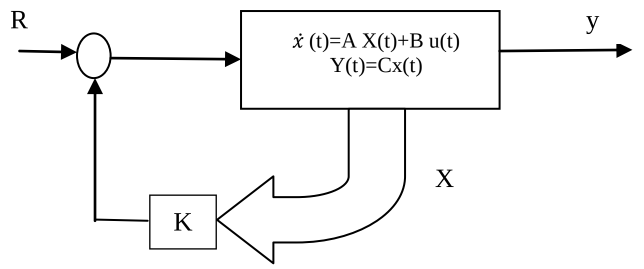


Fig. 2:Full state feed back representation of Inverted pendulum

## B. *Simulation of Inverted Pendulum*

Using the LQR method, the effect of optimal control depends on the selection of weighting matrices Q and R,if Q and R selected not properly, it make the solution cannot meet the actual system performance requirements. In general, Q and R are taken the diagonal matrix, the current approach for selecting weighting matrices Q and R is simulation of trial, after finding a suitable Q and R, it allows the use of computers to find the optimal gain matrix K easily.

According to the state equation which we had got according to straight line of the inverted pendulum system, the three state variables x, $\phi_1$, $\phi_2$, representing the cart displacement, pendulum 1st angle and pendulum 2nd angle. The output is given as follows:

$$y=[x\ \phi_1\ \phi_2]$$

In Matlab statement is K=lqr(A,B,Q,R) [9-10]. The weight matrix Q = diag(1 0 1 0 1 0), diagonal matrix and R=1. In the above command K is the feed back gain values. Whose values are as follow: K = [1.0000 2.3611 -167.9862 -13.3572 185.6691 28.3933]. Step response of the system is shown in fig. 3.

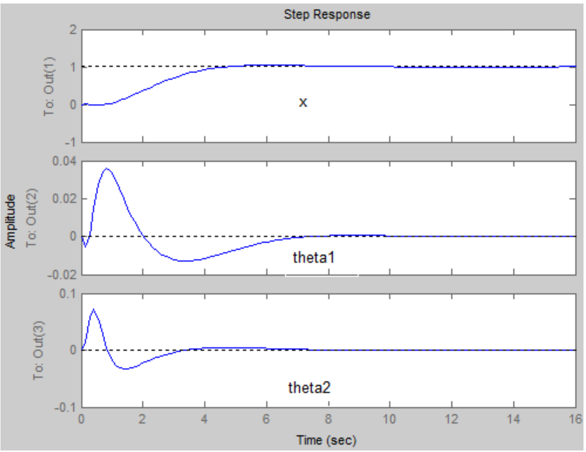


Fig. 3: Step response of the system.

From the figure 3, it is evident that the settling time and rise time is too large while peak overshoot is small.

To minimize the rise time and settling time procedure is as follows: Replace the element of matrix (Q) by Q = diag(100 0 100 0 100 0). By doing so values of feed back gain values (K) will change to as follow: K = [10.0000 12.2699 -209.9083 -13.0043 279.9244 43.4810]. The change in the rise time and settling is evident from the figure 4.

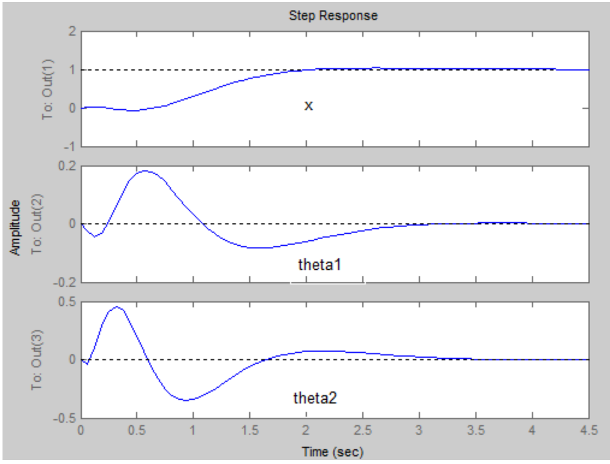


Fig. 4: Improved step response of the system

## V. CONCLUSION

In this paper, an LQR based controller is successfully designed for the double inverted pendulum system. The simulation results for the system for step response are also drawn for the system. In this paper techniques are also discussed in order to reduce the settling and rise time of the system.

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